

The Quaternionic Cauchy–Szegő Kernel on the Quaternionic Siegel Half Space

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Abstract. The quaternionic Cauchy–Szegő kernel of the Hardy space $\mathcal{H}^2(\mathcal{U}_n)$ on the quaternionic Siegel half space \mathcal{U}_n is derived and the Hardy spaces on the octonionic Siegel half space is investigated.

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1 Introduction and main results

The Siegel upper half space in \mathbb{C}^{n+1} is defined by

$$\mathcal{U}^n = \left\{ z \in \mathbb{C}^{n+1} : \operatorname{Im} z_{n+1} > \sum_{j=1}^n |z_j|^2 \right\},$$

the space $\mathcal{H}^2(\mathcal{U}^n)$, consists of all functions F holomorphic on \mathcal{U}^n , for which

$$\sup_{\varepsilon > 0} \int_{\partial \mathcal{U}^n} |F_\varepsilon(z)|^2 d\beta(z) < \infty,$$

where

$$F_\varepsilon(z) = F(z + \varepsilon \mathbf{i}), \quad \mathbf{i} = (\underbrace{0, \dots, 0}_{(2n+1) \text{ 0s}}, i).$$

For any $F \in \mathcal{H}^2(\mathcal{U}^n)$, the boundary limit F^b exists in the sense of almost everywhere and $L^2(\partial \mathcal{U}^n)$. Moreover, we have the integral representation ([10, 18])

$$F(z) = \int_{\partial \mathcal{U}^n} S(z, \omega) F^b(\omega) d\beta(\omega), \quad z \in \mathcal{U}^n,$$

where

$$S(z, \omega) = \frac{n!}{4\pi^{n+1}} \frac{1}{r^{n+1}(z, \omega)}$$

is called the complex Cauchy–Szegő kernel and $r(z, \omega) = \frac{i}{2}(\overline{\omega_{n+1}} - z_{n+1}) - \sum_{k=1}^n z_k \overline{\omega_k}$.

It is natural to ask that whether one could built up an analogous theory in quaternionic analysis, or even in octonionic analysis? As we know, the Heisenberg group of several complex variables can be identified with the boundary of the Siegel upper half space \mathcal{U}^n ([18]), this suggests that we should first investigate the quaternionic Heisenberg group. To our knowledge, the quaternionic Heisenberg

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group was first introduced by Barker and Salamon in their paper [3], and rediscovered in [5], where the Hardy space $\mathcal{H}^2(\mathcal{U}_1)$ of two quaternionic variables was studied. But alas, the Cauchy–Szegő kernel obtained in [5] is not correct. In this paper, we derive the desired quaternionic Cauchy–Szegő kernel of the quaternionic Hardy space $\mathcal{H}^2(\mathcal{U}_n)$ on the quaternionic Siegel half space $\mathcal{U}_n \subset \mathbb{H}^{n+1}$ of several quaternionic variables. The difficulty lies in how to determine the explicit form of the kernel and how to deal with the involved higher order partial derivatives of the quaternionic Cauchy kernel. These obstacles arise because the structure of left (right) \mathbb{H} -regular functions is much more complicated than that of holomorphic functions in complex analysis. For instance, z^n is holomorphic for all $n \in \mathbb{Z}$, but there is no integer n such that q^n is \mathbb{H} -regular, except for $n = 0$. After that, we prove the L^p boundedness of the Cauchy–Szegő projection operator, and give a characterization for the Hardy space consists of octonionic analytic functions on the octonionic Siegel half space, which generalizes the quaternionic Hardy space $\mathcal{H}^p(\mathcal{U}_n)$ to the context of octonions.

The quaternionic Siegel half space in \mathbb{H}^{n+1} ($n \geq 1$) is defined by

$$\mathcal{U}_n = \{(q', q_{n+1}) \in \mathbb{H}^{n+1} : \operatorname{Re} q_{n+1} > |q'|^2\},$$

where $q' = (q_1, q_2, \dots, q_n)$ and $|q'| = (\sum_1^n |q_i|^2)^{\frac{1}{2}}$. To simplify the notations we write $\overline{q'} = (\overline{q_1}, \overline{q_2}, \dots, \overline{q_n})$, $\omega' + q' = \sum_1^n (\omega_i + q_i)$ and $\omega' \cdot q' = \sum_1^n \omega_i q_i$.

For any $\delta > 0$, define the dilation on \mathcal{U}_n by $\delta \circ (q', q_{n+1}) = (\delta q', \delta^2 q_{n+1})$, and for any

$$\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2, \dots, \mathcal{R}_n) \in \mathbb{H}^n \quad (|\mathcal{R}_i| = 1, i = 1, \dots, n),$$

define the rotation on \mathcal{U}_n by $\mathcal{R}(q', q_{n+1}) = (\mathcal{R} \cdot q', q_{n+1})$.

We denote by \mathcal{Q}_n the quaternionic Heisenberg group, viz., $\mathcal{Q}_n = \mathbb{H}^n \times \mathbb{R}^3 = \{[\omega', t] : \omega' \in \mathbb{H}^n, t = (t_1, t_2, t_3) \in \mathbb{R}^3\}$, and define the multiplication by $[\alpha, t] \diamond [\beta, s] = [\alpha + \beta, t_1 + s_1 - 2\operatorname{Im}_1(\overline{\beta} \cdot \alpha), t_2 + s_2 - 2\operatorname{Im}_2(\overline{\beta} \cdot \alpha), t_3 + s_3 - 2\operatorname{Im}_3(\overline{\beta} \cdot \alpha)]$.

For each $h = [\omega', t] \in \mathcal{Q}_n$, define the translation on \mathcal{U}_n by

$$q = (q', q_{n+1}) \mapsto h(q) = (q' + \omega', q_{n+1} + |\omega'|^2 + 2\overline{\omega'} \cdot q' + \mathbf{e} \cdot t),$$

here $\mathbf{e} \cdot t = \sum_{i=1}^3 e_i t_i$, and e_i ($1 \leq i \leq 3$) are three imaginary units of \mathbb{H} (see Section 2).

The three automorphisms of the domain \mathcal{U}_n mentioned above contribute a lot to the calculation of the Cauchy–Szegő kernel.

For $0 < p < \infty$, the Hardy space $\mathcal{H}^p(\mathcal{U}_n)$ consists of all functions $F(q)$ which are left \mathbb{H} -regular on \mathcal{U}_n and satisfy

$$\|F\|_{\mathcal{H}^p(\mathcal{U}_n)} := \left(\sup_{\varepsilon > 0} \int_{\partial \mathcal{U}_n} |F_\varepsilon(q)|^p d\beta(q) \right)^{1/p} < \infty,$$

where

$$F_\varepsilon(q) = F(q + \varepsilon \mathbf{e}_0), \quad \mathbf{e}_0 = \underbrace{(0, \dots, 0)}_{4n \text{ 0s}}, 1, 0, 0, 0),$$

and $d\beta(q)$ can be identified with the Lebesgue measure on $\partial \mathcal{U}_n$.

With these notations, we obtain the following result:

Theorem 1.1. *Suppose $F \in \mathcal{H}^p(\mathcal{U}_n)$ ($\frac{2}{3} < p < \infty$), then*

- (i) *There exists an $F^b \in L^p(\partial \mathcal{U}_n)$, such that $F_\varepsilon(q)|_{\partial \mathcal{U}_n} \rightarrow F^b$ ($\varepsilon \rightarrow 0$) in the sense of $L^p(\partial \mathcal{U}_n)$ norm and almost everywhere.*
- (ii) *$\{F^b\}$ is a closed subspace of $L^p(\partial \mathcal{U}_n)$, and $\|F^b\|_{L^p(\partial \mathcal{U}_n)} = \|F\|_{\mathcal{H}^p(\mathcal{U}_n)}$.*

(iii) If $F \in \mathcal{H}^2(\mathcal{U}_n)$, then

$$F(q) = \int_{\partial\mathcal{U}_n} S(q, \omega) F^b(\omega) d\beta(\omega), \quad q = (q', q_{n+1}) \in \mathcal{U}_n,$$

where

$$S(q, \omega) = s(q_{n+1} + \overline{\omega_{n+1}} - 2\overline{\omega'} \cdot q')$$

is called the quaternionic Cauchy–Szegő kernel and

$$s(\nu) = \left(\frac{2}{\pi}\right)^{2n} \frac{\partial^{2n}}{\partial x_0^{2n}} E(\nu),$$

$$E(\nu) = \frac{1}{\sigma_4} \frac{\overline{\nu}}{|\nu|^4} = \frac{1}{2\pi^2} \frac{\overline{\nu}}{|\nu|^4}, \quad \nu = \sum_{i=0}^3 x_i e_i \in \mathbb{H}.$$

If we use \mathbb{A} to denote the complex field \mathbb{C} or the skew field \mathbb{H} , $m = 2$ or 4 is the dimension of \mathbb{A} over \mathbb{R} , then the Siegel half space in \mathbb{A}^{n+1} can be written uniformly as $\mathcal{D}_n = \{(\zeta', \zeta_{n+1}) \in \mathbb{A}^{n+1} : \operatorname{Re} \zeta_{n+1} > |\zeta'|^2\}$. Similarly, we can define the function spaces $\mathcal{H}^p(\mathcal{D}_n)$, thus we can write the complex-valued and the quaternion-valued Cauchy–Szegő kernels in one form:

Theorem 1.2. For every $F \in \mathcal{H}^2(\mathcal{D}_n)$, we can represent F by the integral formula

$$F(\zeta) = \int_{\partial\mathcal{D}_n} S(\zeta, \omega) F^b(\omega) d\beta(\omega), \quad \zeta = (\zeta', \zeta_{n+1}) \in \mathcal{D}_n,$$

where

$$S(\zeta, \omega) = s(\zeta_{n+1} + \overline{\omega_{n+1}} - 2\overline{\omega'} \cdot \zeta'),$$

$$s(\nu) = \left(-\frac{2}{\pi}\right)^{\frac{mn}{2}} \frac{\partial^{\frac{mn}{2}}}{\partial x_0^{\frac{mn}{2}}} E(\nu),$$

$$E(\nu) = \frac{1}{\sigma_m} \frac{\overline{\nu}}{|\nu|^m} = \frac{\Gamma(\frac{m}{2})}{2\pi^{\frac{m}{2}}} \frac{\overline{\nu}}{|\nu|^m}, \quad \nu = \sum_i x_i e_i \in \mathbb{A}.$$

The rest of this paper is organized as follows: Section 2 contains some basic knowledge of quaternion algebra, Cayley algebra and respectively, their analysis. Section 3 is mainly devoted to the proof of Theorem 1.1. In the last section, we discuss the octonionic Siegel half space \mathcal{U} , the octonionic Heisenberg group \mathcal{O} and the octonionic Hardy spaces $\mathcal{H}^p(\mathcal{U})$.

2 Preliminaries

If an algebra \mathbb{A} is meanwhile a normed vector space, and its norm “ $\|\cdot\|$ ” satisfies $\|ab\| = \|a\|\|b\|$, then we call \mathbb{A} a normed algebra. If $ab = 0$ ($a, b \in \mathbb{A}$) implies $a = 0$ or $b = 0$, then we call \mathbb{A} a division algebra. Early in 1898, Hurwitz had proved that the real numbers \mathbb{R} , complex numbers \mathbb{C} , quaternions \mathbb{H} and octonions \mathbb{O} are the only normed division algebras over \mathbb{R} ([11]), with the imbedding relation $\mathbb{R} \subseteq \mathbb{C} \subseteq \mathbb{H} \subseteq \mathbb{O}$.

Any quaternion $q \in \mathbb{H}$ is of the form $q = \sum_{i=0}^3 x_i e_i$ (we often omit the algebraic unit element e_0 and write $q = x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3$), where $\operatorname{Re} q = x_0$, $\operatorname{Im}_i q = x_i$ ($1 \leq i \leq 3$) all belong to \mathbb{R} and the basis e_i ($0 \leq i \leq 3$) satisfies $e_0^2 = e_0$, $e_i e_0 = e_0 e_i = e_i$, $e_i^2 = -1$ ($1 \leq i \leq 3$) and

$$e_1 e_2 = e_3 = -e_2 e_1, \quad e_2 e_3 = e_1 = -e_3 e_2, \quad e_3 e_1 = e_2 = -e_1 e_3.$$

$|q| = (\sum_0^3 x_i^2)^{\frac{1}{2}}$ is the norm of q , and $\bar{q} = x_0 e_0 - \sum_{i=1}^3 x_i e_i$ is the conjugate of q . The inverse of q ($q \neq 0$) is given by $q^{-1} = \bar{q}/|q|^2$, due to $q\bar{q} = \bar{q}q = |q|^2$. For any $q_1, q_2, q_3 \in \mathbb{H}$, we have $|q_1 q_2| = |q_1||q_2|$, $\overline{q_1 q_2} = \bar{q}_2 \bar{q}_1$ and $(q_1 q_2) q_3 = q_1 (q_2 q_3)$. With respect to the multiplication law, quaternions \mathbb{H} is associative but non-commutative.

A function $f \in C^1(\Omega, \mathbb{H})$ is said to be left (right) \mathbb{H} -regular in the open set $\Omega \subset \mathbb{R}^4$ if and only if

$$Df = e_0 \frac{\partial f}{\partial x_0} + e_1 \frac{\partial f}{\partial x_1} + e_2 \frac{\partial f}{\partial x_2} + e_3 \frac{\partial f}{\partial x_3} = 0$$

$$\left(fD = \frac{\partial f}{\partial x_0} e_0 + \frac{\partial f}{\partial x_1} e_1 + \frac{\partial f}{\partial x_2} e_2 + \frac{\partial f}{\partial x_3} e_3 = 0 \right).$$

If a left \mathbb{H} -regular function is meanwhile right \mathbb{H} -regular, then we call it an \mathbb{H} -regular function.

The study of quaternionic analysis was started from 1930s ([7, 8, 20]), now it becomes more and more important in physics and engineering.

As the largest normed division algebra, octonion is discovered by John T. Graves in 1843, and then by Arthur Cayley in 1845 independently, which is sometimes referred to as Cayley number or the Cayley algebra, it is an 8 dimensional division algebra over \mathbb{R} with the basis e_0, e_1, \dots, e_7 satisfying $e_0^2 = e_0$, $e_i e_0 = e_0 e_i = e_i$, $e_i^2 = -1$, $i = 1, 2, \dots, 7$. Denote

$$W = \{(1, 2, 3), (1, 4, 5), (1, 7, 6), (2, 4, 6), (2, 5, 7), (3, 4, 7), (3, 6, 5)\},$$

then any triple $(\alpha, \beta, \gamma) \in W$ obeys

$$e_\alpha e_\beta = e_\gamma = -e_\beta e_\alpha, \quad e_\beta e_\gamma = e_\alpha = -e_\gamma e_\beta, \quad e_\gamma e_\alpha = e_\beta = -e_\alpha e_\gamma,$$

which completely determine the multiplication of octonions by linearity.

For any $x = \sum_0^7 x_i e_i \in \mathbb{O}$, $\text{Re } x = x_0$ is called the scalar (real) part of x and $\vec{x} = \sum_1^7 x_i e_i$ is called its vector part. The i th component x_i is denoted by $\text{Im}_i x$ ($1 \leq i \leq 7$). Define $\bar{x} = \sum_0^7 x_i \bar{e}_i = x_0 - \vec{x}$ as the conjugate of x , and $|x| = (\sum_0^7 x_i^2)^{\frac{1}{2}}$ as the norm (modulus) of x , they satisfy: $|xy| = |x||y|$, $x\bar{x} = \bar{x}x = |x|^2$, $\overline{xy} = \bar{y}\bar{x}$ ($x, y \in \mathbb{O}$). If $x \neq 0$, $x^{-1} = \bar{x}/|x|^2$ gives the inverse of x .

Octonionic multiplication is neither commutative nor associative. $[x, y, z] = (xy)z - x(yz)$ is called the associator of $x, y, z \in \mathbb{O}$, which satisfies

$$[x, y, z] = [y, z, x] = -[y, x, z], \quad [x, x, y] = 0 = [\bar{x}, x, y].$$

The octonionic analysis, which is a generalization of quaternionic analysis to higher dimensions, was studied systematically since 1995 ([12]). Suppose Ω is an open subset of \mathbb{R}^8 , if $f \in C^1(\Omega, \mathbb{O})$ satisfies $Df = \sum_0^7 e_i \frac{\partial f}{\partial x_i} = 0$ ($fD = \sum_0^7 \frac{\partial f}{\partial x_i} e_i = 0$), then f is said to be left (right) \mathbb{O} -analytic in Ω , here the Dirac operator D and its conjugate \bar{D} are defined by $D = \sum_0^7 e_i \frac{\partial}{\partial x_i}$ and $\bar{D} = \sum_0^7 \bar{e}_i \frac{\partial}{\partial x_i}$ respectively. A function f is \mathbb{O} -analytic means that f is left \mathbb{O} -analytic, and also right \mathbb{O} -analytic. From $\bar{D}(Df) = (\bar{D}D)f = \Delta f = f(D\bar{D}) = (fD)\bar{D}$, we know that any left (right) \mathbb{O} -analytic (\mathbb{H} -regular) function is always harmonic.

In order to build up the theory of H^p -spaces in higher-dimensional Euclidean spaces, in 1960, E. M. Stein and G. Weiss generalized the notion of holomorphic functions to the system of conjugate harmonic functions ([19]), which is now called the Stein–Weiss conjugate harmonic system, it is a vector of harmonic functions $(\mu_0, \mu_1, \dots, \mu_n)$ of variables (x_0, x_1, \dots, x_n) , whose components satisfy the following generalized Cauchy–Riemann equations:

$$\begin{cases} \sum_{i=0}^n \frac{\partial \mu_i}{\partial x_i} = 0, \\ \frac{\partial \mu_j}{\partial x_k} = \frac{\partial \mu_k}{\partial x_j}, \quad 0 \leq j < k \leq n. \end{cases}$$

If $F(x_0, x_1, \dots, x_7) = (f_0, f_1, \dots, f_7)$ is a Stein–Weiss conjugate harmonic system in Ω , then $\overline{F} = f_0 - \sum_{i=1}^7 f_i e_i$ is an \mathbb{O} -analytic function ([12]). But inversely, this is not true ([16]). For more information and recent progress about octonionic analysis, we refer the reader to [2, 12–17].

3 The proof of Theorem 1.1 and the Cauchy–Szegő projection operator

Proof of Theorem 1.1. By analogous discussions as in [18], we can easily prove (i) and (ii). As to (iii), by the same method in the case of several complex variables, we can show the existence and uniqueness of $S(q, \omega)$, and $S(q, \omega) = \overline{S(\omega, q)}$. The proofs are omitted here.

Invoking the three automorphisms of the domain \mathcal{U}_n mentioned in Section 1, we can derive the following integral identities:

$$\begin{aligned} F(q) &= \int_{\partial \mathcal{U}_n} S(\delta \circ q, \delta \circ \omega) \delta^{4n+6} F^b(\omega) d\beta(\omega), & \forall \delta > 0, \\ F(q) &= \int_{\partial \mathcal{U}_n} S(\mathcal{R}(q), \mathcal{R}(\omega)) F^b(\omega) d\beta(\omega), & \forall \mathcal{R}, \\ F(q) &= \int_{\partial \mathcal{U}_n} S(h(q), h(\omega)) F^b(\omega) d\beta(\omega), & \forall h \in \mathcal{Q}_n. \end{aligned}$$

These imply that

$$S(q, \omega) = S(\delta \circ q, \delta \circ \omega) \delta^{4n+6} = S(\mathcal{R}(q), \mathcal{R}(\omega)) = S(h(q), h(\omega)).$$

Hence, $s(q_{n+1}) := S(q, 0)$ is independent of q' and satisfies

$$S(q, \omega) = s(q_{n+1} + \overline{\omega_{n+1}} - 2\overline{\omega'} \cdot q').$$

Further more, $s(q_{n+1})$ is left \mathbb{H} -regular with respect to one quaternionic variable q_{n+1} and homogeneous of negative order $-2n - 3$. Similar to the method in [4], we can show that

$$\left\{ \frac{\partial^{2n}}{\partial x_0^r \partial x_1^s \partial x_2^t} E(\nu) : r + s + t = 2n \right\}$$

forms a basis of this homogeneous class, where $E(\nu) = \frac{1}{2\pi^2} \frac{\overline{\nu}}{|\nu|^4}$ ($\nu = \sum_{i=0}^3 x_i e_i \in \mathbb{H}$) is called the quaternionic Cauchy kernel which is \mathbb{H} -regular in $\mathbb{R}^4 \setminus \{0\}$. So the function $s(\cdot)$ must be of the form

$$s(\nu) = \sum_{r+s+t=2n} \frac{\partial^{2n} E(\nu)}{\partial x_0^r \partial x_1^s \partial x_2^t} c_{r,s,t}.$$

The rest of the proof will be devoted to the determination of the coefficients $c_{r,s,t} \in \mathbb{H}$.

For this purpose, it is convenient for the readers that we give an outline here. That is, first, we build up a linear system of infinite equations that contains the undetermined coefficients by the special value method, and then solve the system. Here we should point out that in the case of several complex variables, to determine the complex Cauchy–Szegő kernel, we just need to compute one unknown coefficient, and the computation is direct and easy. While in the quaternionic case, there are $(n+1)(2n+1)$ unknown coefficients to compute, due to the dimension of the space of homogeneous left \mathbb{H} -regular functions of negative order $-(2n+3)$ is $C_{2n+2}^2 = (n+1)(2n+1)$. Also, in such case, the higher order partial derivatives of the quaternionic Cauchy kernel $E(\nu)$ are not easy to handle with, unless n is specific and small. So we would use some different techniques to deal with it. To be specific, we calculate the multi-integrals by Fourier transform and spherical coordinates transform. While solving the system, we adopt the limiting arguments.

In what follows we use N to denote the Newton potential $\frac{1}{|\nu|^2} = \frac{1}{x_0^2 + x_1^2 + x_2^2 + x_3^2}$ in \mathbb{R}^4 . We start by proving two basic properties concerning the multi-integrals.

Proposition 3.1. For any $p_0 + p_1 + p_2 + p_3 = \alpha$ and $q_0 + q_1 + q_2 + q_3 = \gamma$, we have

$$\begin{aligned} & \int_{\mathbb{R}^3} \frac{\partial^\alpha N}{\partial x_0^{p_0} \partial x_1^{p_1} \partial x_2^{p_2} \partial x_3^{p_3}} \frac{\partial^\gamma N}{\partial x_0^{q_0} \partial x_1^{q_1} \partial x_2^{q_2} \partial x_3^{q_3}} dx_1 dx_2 dx_3 \\ &= \int_{\mathbb{R}^3} 2^{\alpha+\gamma} \pi^{\alpha+\gamma+2} (-1)^{p_0+\gamma} i^{\alpha+\gamma-p_0-q_0} (x_1^2 + x_2^2 + x_3^2)^{\frac{p_0+q_0}{2}-1} \\ & \quad \times x_1^{p_1+q_1} x_2^{p_2+q_2} x_3^{p_3+q_3} e^{-4\pi x_0 \sqrt{x_1^2+x_2^2+x_3^2}} dx_1 dx_2 dx_3. \end{aligned}$$

Proof. By Parseval's theorem,

$$\begin{aligned} & \int_{\mathbb{R}^3} \frac{\partial^\alpha N}{\partial x_0^{p_0} \partial x_1^{p_1} \partial x_2^{p_2} \partial x_3^{p_3}} \frac{\partial^\gamma N}{\partial x_0^{q_0} \partial x_1^{q_1} \partial x_2^{q_2} \partial x_3^{q_3}} dx_1 dx_2 dx_3 \\ &= \int_{\mathbb{R}^3} \left(\frac{\partial^\alpha N}{\partial x_0^{p_0} \partial x_1^{p_1} \partial x_2^{p_2} \partial x_3^{p_3}} \right)^\wedge(x_1, x_2, x_3) \overline{\left(\frac{\partial^\gamma N}{\partial x_0^{q_0} \partial x_1^{q_1} \partial x_2^{q_2} \partial x_3^{q_3}} \right)^\wedge(x_1, x_2, x_3)} dx_1 dx_2 dx_3 \\ &= \int_{\mathbb{R}^3} (2\pi i)^{p_1+p_2+p_3} x_1^{p_1} x_2^{p_2} x_3^{p_3} \frac{\partial^{p_0} \widehat{N}}{\partial x_0^{p_0}} (-2\pi i)^{q_1+q_2+q_3} x_1^{q_1} x_2^{q_2} x_3^{q_3} \overline{\frac{\partial^{q_0} \widehat{N}}{\partial x_0^{q_0}}} dx_1 dx_2 dx_3, \end{aligned}$$

and

$$\begin{aligned} \widehat{N}(x_1, x_2, x_3) &= \int_{\mathbb{R}^3} \frac{e^{-2\pi i(x_1 \xi_1 + x_2 \xi_2 + x_3 \xi_3)}}{x_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2} d\xi \\ &= \int_{\mathbb{R}^3} \frac{e^{-2\pi i \xi_1 \sqrt{x_1^2 + x_2^2 + x_3^2}}}{x_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2} d\xi \\ &= \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} \frac{\cos(2\pi \xi_1 \sqrt{x_1^2 + x_2^2 + x_3^2})}{x_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2} d\xi_1 \right) d\xi_2 d\xi_3 \\ &= \int_{\mathbb{R}^2} \frac{\pi}{\sqrt{x_0^2 + \xi_2^2 + \xi_3^2}} e^{-2\pi \sqrt{(x_1^2 + x_2^2 + x_3^2)(x_0^2 + \xi_2^2 + \xi_3^2)}} d\xi_2 d\xi_3 \\ &= \frac{\pi}{\sqrt{x_1^2 + x_2^2 + x_3^2}} e^{-2\pi x_0 \sqrt{x_1^2 + x_2^2 + x_3^2}}, \end{aligned}$$

the proposition follows. □

Proposition 3.2. Suppose $a > 0$, denote $l = l_0 + l_1 + l_2 + l_3$, we have

$$\begin{aligned} & \int_{\mathbb{R}^3} (x_1^2 + x_2^2 + x_3^2)^{\frac{l_0}{2}} x_1^{l_1} x_2^{l_2} x_3^{l_3} e^{-a\sqrt{x_1^2+x_2^2+x_3^2}} dx_1 dx_2 dx_3 \\ &= \begin{cases} 2a^{-l-3} \Gamma(l+3) \frac{\Gamma(k_1 + \frac{1}{2}) \Gamma(k_2 + \frac{1}{2}) \Gamma(k_3 + \frac{1}{2})}{\Gamma(k_1 + k_2 + k_3 + \frac{3}{2})}, & \text{if } l_1 = 2k_1, l_2 = 2k_2, l_3 = 2k_3, \\ 0, & \text{else.} \end{cases} \end{aligned}$$

Proof. Invoking the spherical coordinates, when $l_1 = 2k_1, l_2 = 2k_2, l_3 = 2k_3$, we have

$$\begin{aligned} & \int_{\mathbb{R}^3} (x_1^2 + x_2^2 + x_3^2)^{\frac{l_0}{2}} x_1^{l_1} x_2^{l_2} x_3^{l_3} e^{-a\sqrt{x_1^2+x_2^2+x_3^2}} dx_1 dx_2 dx_3 \\ &= \int_0^\infty r^{l+2} e^{-ar} dr \int_0^\pi \cos^{2k_1} \theta \sin^{2k_2+2k_3+1} \theta d\theta \int_0^{2\pi} \cos^{2k_2} \varphi \sin^{2k_3} \varphi d\varphi \\ &= a^{-l-3} \Gamma(l+3) \frac{\Gamma(k_1 + \frac{1}{2}) \Gamma(k_2 + k_3 + 1)}{\Gamma(k_1 + k_2 + k_3 + \frac{3}{2})} \frac{2\Gamma(k_2 + \frac{1}{2}) \Gamma(k_3 + \frac{1}{2})}{\Gamma(k_2 + k_3 + 1)} \\ &= 2a^{-l-3} \Gamma(l+3) \frac{\Gamma(k_1 + \frac{1}{2}) \Gamma(k_2 + \frac{1}{2}) \Gamma(k_3 + \frac{1}{2})}{\Gamma(k_1 + k_2 + k_3 + \frac{3}{2})}. \end{aligned}$$

Otherwise, if, for example, l_1 is odd, then the integrand is odd with respect to the variable x_1 , so the integral vanishes. \square

Now we turn to the calculation of the coefficients. First we note that

$$E(\nu) = -\frac{1}{4\pi^2} \bar{D}N = -\frac{1}{4\pi^2} \left(\frac{\partial N}{\partial x_0} - \frac{\partial N}{\partial x_1} e_1 - \frac{\partial N}{\partial x_2} e_2 - \frac{\partial N}{\partial x_3} e_3 \right),$$

we may rewrite $s(\nu)$ as

$$s(\nu) = \sum_{s_0+s_1+s_2=2n} \left(\frac{\partial^{2n+1} N}{\partial x_0^{s_0+1} \partial x_1^{s_1} \partial x_2^{s_2}} - \frac{\partial^{2n+1} N}{\partial x_0^{s_0} \partial x_1^{s_1+1} \partial x_2^{s_2}} e_1 - \frac{\partial^{2n+1} N}{\partial x_0^{s_0} \partial x_1^{s_1} \partial x_2^{s_2+1}} e_2 - \frac{\partial^{2n+1} N}{\partial x_0^{s_0} \partial x_1^{s_1} \partial x_2^{s_2} \partial x_3} e_3 \right) c(s_0, s_1, s_2),$$

where

$$c(s_0, s_1, s_2) = c_0(s_0, s_1, s_2) + c_1(s_0, s_1, s_2) e_1 + c_2(s_0, s_1, s_2) e_2 + c_3(s_0, s_1, s_2) e_3$$

are to be determined. Set

$$F_\lambda(q', q_{n+1}) = \left(\frac{\partial^{\lambda+1} N}{\partial x_0^{t_0+1} \partial x_1^{t_1} \partial x_2^{t_2} \partial x_3^{t_3}} - \frac{\partial^{\lambda+1} N}{\partial x_0^{t_0} \partial x_1^{t_1+1} \partial x_2^{t_2} \partial x_3^{t_3}} e_1 - \frac{\partial^{\lambda+1} N}{\partial x_0^{t_0} \partial x_1^{t_1} \partial x_2^{t_2+1} \partial x_3^{t_3}} e_2 - \frac{\partial^{\lambda+1} N}{\partial x_0^{t_0} \partial x_1^{t_1} \partial x_2^{t_2} \partial x_3^{t_3+1}} e_3 \right) \Big|_{\nu=1+q_{n+1}},$$

where $\lambda = t_0 + t_1 + t_2 + t_3$, then one can verify that $F_\lambda(q', q_{n+1}) \in \mathcal{H}^2(\mathcal{U}_n)$ whenever $\lambda > (2n-3)/2$. By the integral representation, we have

$$\begin{aligned} F_\lambda(0, 1) &= \int_{\partial \mathcal{U}_n} S(q, \omega)|_{q=(0,1)} F_\lambda^b(\omega) d\beta(\omega) = \int_{\partial \mathcal{U}_n} \overline{s(1+\omega_{n+1})} F_\lambda(\omega) d\beta(\omega) \\ &= \sum_{s_0+s_1+s_2=2n} \left(c_0(s_0, s_1, s_2) - c_1(s_0, s_1, s_2) e_1 - c_2(s_0, s_1, s_2) e_2 - c_3(s_0, s_1, s_2) e_3 \right) \\ &\quad \times \left(\int_{\operatorname{Re} \omega_{n+1}=|\omega'|^2} \left(\frac{\partial^{2n+1} N}{\partial x_0^{s_0+1} \partial x_1^{s_1} \partial x_2^{s_2}} \frac{\partial^{\lambda+1} N}{\partial x_0^{t_0+1} \partial x_1^{t_1} \partial x_2^{t_2} \partial x_3^{t_3}} + \frac{\partial^{2n+1} N}{\partial x_0^{s_0} \partial x_1^{s_1+1} \partial x_2^{s_2}} \frac{\partial^{\lambda+1} N}{\partial x_0^{t_0} \partial x_1^{t_1+1} \partial x_2^{t_2} \partial x_3^{t_3}} \right. \right. \\ &\quad \left. \left. + \frac{\partial^{2n+1} N}{\partial x_0^{s_0} \partial x_1^{s_1} \partial x_2^{s_2+1} \partial x_3} \frac{\partial^{\lambda+1} N}{\partial x_0^{t_0} \partial x_1^{t_1} \partial x_2^{t_2+1} \partial x_3^{t_3}} + \frac{\partial^{2n+1} N}{\partial x_0^{s_0} \partial x_1^{s_1} \partial x_2^{s_2} \partial x_3} \frac{\partial^{\lambda+1} N}{\partial x_0^{t_0} \partial x_1^{t_1} \partial x_2^{t_2} \partial x_3^{t_3+1}} \right) \Big|_{\nu=1+\omega_{n+1}} d\beta(\omega) \right. \\ &\quad + \int_{\operatorname{Re} \omega_{n+1}=|\omega'|^2} \left(- \frac{\partial^{2n+1} N}{\partial x_0^{s_0+1} \partial x_1^{s_1} \partial x_2^{s_2}} \frac{\partial^{\lambda+1} N}{\partial x_0^{t_0} \partial x_1^{t_1+1} \partial x_2^{t_2} \partial x_3^{t_3}} + \frac{\partial^{2n+1} N}{\partial x_0^{s_0} \partial x_1^{s_1+1} \partial x_2^{s_2}} \frac{\partial^{\lambda+1} N}{\partial x_0^{t_0+1} \partial x_1^{t_1} \partial x_2^{t_2} \partial x_3^{t_3}} \right. \\ &\quad \left. \left. + \frac{\partial^{2n+1} N}{\partial x_0^{s_0} \partial x_1^{s_1} \partial x_2^{s_2+1} \partial x_3} \frac{\partial^{\lambda+1} N}{\partial x_0^{t_0} \partial x_1^{t_1} \partial x_2^{t_2+1} \partial x_3^{t_3}} - \frac{\partial^{2n+1} N}{\partial x_0^{s_0} \partial x_1^{s_1} \partial x_2^{s_2+1} \partial x_3} \frac{\partial^{\lambda+1} N}{\partial x_0^{t_0} \partial x_1^{t_1} \partial x_2^{t_2} \partial x_3^{t_3+1}} \right) \Big|_{\nu=1+\omega_{n+1}} d\beta(\omega) e_1 \right. \\ &\quad \left. + \int_{\operatorname{Re} \omega_{n+1}=|\omega'|^2} \left(- \frac{\partial^{2n+1} N}{\partial x_0^{s_0+1} \partial x_1^{s_1} \partial x_2^{s_2}} \frac{\partial^{\lambda+1} N}{\partial x_0^{t_0} \partial x_1^{t_1} \partial x_2^{t_2+1} \partial x_3^{t_3}} + \frac{\partial^{2n+1} N}{\partial x_0^{s_0} \partial x_1^{s_1} \partial x_2^{s_2+1} \partial x_3} \frac{\partial^{\lambda+1} N}{\partial x_0^{t_0+1} \partial x_1^{t_1} \partial x_2^{t_2} \partial x_3^{t_3}} \right. \right. \\ &\quad \left. \left. + \frac{\partial^{2n+1} N}{\partial x_0^{s_0} \partial x_1^{s_1+1} \partial x_2^{s_2}} \frac{\partial^{\lambda+1} N}{\partial x_0^{t_0} \partial x_1^{t_1} \partial x_2^{t_2} \partial x_3^{t_3+1}} - \frac{\partial^{2n+1} N}{\partial x_0^{s_0} \partial x_1^{s_1} \partial x_2^{s_2} \partial x_3} \frac{\partial^{\lambda+1} N}{\partial x_0^{t_0} \partial x_1^{t_1+1} \partial x_2^{t_2} \partial x_3^{t_3}} \right) \Big|_{\nu=1+\omega_{n+1}} d\beta(\omega) e_2 \right) \end{aligned}$$

$$+ \int_{\operatorname{Re} \omega_{n+1}=|\omega'|^2} \left(- \frac{\partial^{2n+1} N}{\partial x_0^{s_0+1} \partial x_1^{s_1} \partial x_2^{s_2}} \frac{\partial^{\lambda+1} N}{\partial x_0^{t_0} \partial x_1^{t_1} \partial x_2^{t_2} \partial x_3^{t_3+1}} + \frac{\partial^{2n+1} N}{\partial x_0^{s_0} \partial x_1^{s_1} \partial x_2^{s_2} \partial x_3} \frac{\partial^{\lambda+1} N}{\partial x_0^{t_0+1} \partial x_1^{t_1} \partial x_2^{t_2} \partial x_3^{t_3}} \right. \\ \left. + \frac{\partial^{2n+1} N}{\partial x_0^{s_0} \partial x_1^{s_1} \partial x_2^{s_2+1}} \frac{\partial^{\lambda+1} N}{\partial x_0^{t_0} \partial x_1^{t_1+1} \partial x_2^{t_2} \partial x_3^{t_3}} - \frac{\partial^{2n+1} N}{\partial x_0^{s_0} \partial x_1^{s_1+1} \partial x_2^{s_2}} \frac{\partial^{\lambda+1} N}{\partial x_0^{t_0} \partial x_1^{t_1} \partial x_2^{t_2+1} \partial x_3^{t_3}} \right) \Big|_{\nu=1+\omega_{n+1}} d\beta(\omega) e_3 \Big).$$

By Proposition 3.1,

$$F_\lambda(0, 1) = \sum_{s_0+s_1+s_2=2n} \left(c_0(s_0, s_1, s_2) - c_1(s_0, s_1, s_2)e_1 - c_2(s_0, s_1, s_2)e_2 - c_3(s_0, s_1, s_2)e_3 \right) \\ \times \left(\int_{\mathbb{H}^n} \left(\int_{\mathbb{R}^3} 2^{2n+\lambda+3} \pi^{2n+\lambda+4} (-1)^{s_0+\lambda} i^{2n+\lambda-s_0-t_0} (x_1^2 + x_2^2 + x_3^2)^{\frac{s_0+t_0}{2}} \right. \right. \\ \left. \left. \times x_1^{s_1+t_1} x_2^{s_2+t_2} x_3^{t_3} e^{-4\pi x_0 \sqrt{x_1^2+x_2^2+x_3^2}} dx_1 dx_2 dx_3 \right) \Big|_{x_0=1+|\omega'|^2} d\omega' \right. \\ + \int_{\mathbb{H}^n} \left(\int_{\mathbb{R}^3} 2^{2n+\lambda+3} \pi^{2n+\lambda+4} (-1)^{s_0+\lambda+1} i^{2n+\lambda+1-s_0-t_0} (x_1^2 + x_2^2 + x_3^2)^{\frac{s_0+t_0-1}{2}} \right. \\ \left. \times x_1^{s_1+t_1+1} x_2^{s_2+t_2} x_3^{t_3} e^{-4\pi x_0 \sqrt{x_1^2+x_2^2+x_3^2}} dx_1 dx_2 dx_3 \right) \Big|_{x_0=1+|\omega'|^2} d\omega' e_1 \\ + \int_{\mathbb{H}^n} \left(\int_{\mathbb{R}^3} 2^{2n+\lambda+3} \pi^{2n+\lambda+4} (-1)^{s_0+\lambda+1} i^{2n+\lambda+1-s_0-t_0} (x_1^2 + x_2^2 + x_3^2)^{\frac{s_0+t_0-1}{2}} \right. \\ \left. \times x_1^{s_1+t_1} x_2^{s_2+t_2+1} x_3^{t_3} e^{-4\pi x_0 \sqrt{x_1^2+x_2^2+x_3^2}} dx_1 dx_2 dx_3 \right) \Big|_{x_0=1+|\omega'|^2} d\omega' e_2 \\ + \int_{\mathbb{H}^n} \left(\int_{\mathbb{R}^3} 2^{2n+\lambda+3} \pi^{2n+\lambda+4} (-1)^{s_0+\lambda+1} i^{2n+\lambda+1-s_0-t_0} (x_1^2 + x_2^2 + x_3^2)^{\frac{s_0+t_0-1}{2}} \right. \\ \left. \times x_1^{s_1+t_1} x_2^{s_2+t_2} x_3^{t_3+1} e^{-4\pi x_0 \sqrt{x_1^2+x_2^2+x_3^2}} dx_1 dx_2 dx_3 \right) \Big|_{x_0=1+|\omega'|^2} d\omega' e_3 \Big). \quad (3.1)$$

When t_i ($0 \leq i \leq 3$) vary, we will get a system of linear equations in variables $c_i(s_0, s_1, s_2)$ ($0 \leq i \leq 3, s_0 + s_1 + s_2 = 2n$). Now, we solve this system.

Taking $t_1 = 2q_1$, $t_2 = 2q_2$, $t_3 = 2q_3 + 1$, one can prove by induction that

$$F_\lambda(0, 1) = (-1)^{t_0+q_1+q_2+q_3} 2^{-\lambda} \frac{\Gamma(\lambda+3)\Gamma(q_1+q_2+q_3+2)\Gamma(2q_1)\Gamma(2q_2)\Gamma(2q_3+2)}{\Gamma(2q_1+2q_2+2q_3+4)\Gamma(q_1)\Gamma(q_2)\Gamma(q_3+1)} e_3 \\ = (-1)^{t_0+q_1+q_2+q_3} 2^{-\lambda-4} \pi^{-1} \frac{\Gamma(\lambda+3)\Gamma(q_1+\frac{1}{2})\Gamma(q_2+\frac{1}{2})\Gamma(q_3+\frac{3}{2})}{\Gamma(q_1+q_2+q_3+\frac{5}{2})} e_3,$$

here the formula $\Gamma(x)\Gamma(x+\frac{1}{2}) = 2^{1-2x}\Gamma(\frac{1}{2})\Gamma(2x)$ is used. Applying Proposition 3.2 to the right hand side of system (3.1), we get

$$\sum_{p_0+p_1+p_2=n} \left(c_0(2p_0, 2p_1, 2p_2) - c_1(2p_0, 2p_1, 2p_2)e_1 - c_2(2p_0, 2p_1, 2p_2)e_2 - c_3(2p_0, 2p_1, 2p_2)e_3 \right) \\ \times (-1)^{t_0+q_1+q_2+q_3+n+p_0+1} 2^{-2n-\lambda-2} \pi^{2n+1} \frac{\Gamma(\lambda+3)\Gamma(p_1+q_1+\frac{1}{2})\Gamma(p_2+q_2+\frac{1}{2})\Gamma(q_3+\frac{3}{2})}{\Gamma(q_1+q_2+q_3+n-p_0+\frac{5}{2})} e_3 \\ = F_\lambda(0, 1) = (-1)^{t_0+q_1+q_2+q_3} 2^{-\lambda-4} \pi^{-1} \frac{\Gamma(\lambda+3)\Gamma(q_1+\frac{1}{2})\Gamma(q_2+\frac{1}{2})\Gamma(q_3+\frac{3}{2})}{\Gamma(q_1+q_2+q_3+\frac{5}{2})} e_3. \quad (3.2)$$

Hence,

$$\sum_{p_0+p_1+p_2=n} (-1)^{p_0} \frac{\Gamma(p_1+q_1+\frac{1}{2})\Gamma(p_2+q_2+\frac{1}{2})}{\Gamma(q_1+q_2+q_3+n-p_0+\frac{5}{2})} c_i(2p_0, 2p_1, 2p_2) = 0, \quad 1 \leq i \leq 3, \quad (3.3)$$

and

$$\sum_{p_0+p_1+p_2=n} (-1)^{n+p_0+1} \frac{\Gamma(p_1+q_1+\frac{1}{2})\Gamma(p_2+q_2+\frac{1}{2})}{\Gamma(q_1+q_2+q_3+n-p_0+\frac{5}{2})} c_0(2p_0, 2p_1, 2p_2) = \frac{2^{2n-2}}{\pi^{2n+2}} \frac{\Gamma(q_1+\frac{1}{2})\Gamma(q_2+\frac{1}{2})}{\Gamma(q_1+q_2+q_3+\frac{5}{2})}. \quad (3.4)$$

While taking $t_1 = 2q_1 + 1$, $t_2 = 2q_2$, $t_3 = 2q_3 + 1$, we will get

$$\begin{aligned} & \sum_{p_0+p_1+p_2=n-1} \frac{c(2p_0+1, 2p_1+1, 2p_2)}{(-1)^{t_0+q_1+q_2+q_3+n+p_0+1} 2^{-2n-\lambda-2} \pi^{2n+1}} \\ & \times \frac{\Gamma(\lambda+3)\Gamma(p_1+q_1+\frac{3}{2})\Gamma(p_2+q_2+\frac{1}{2})\Gamma(q_3+\frac{3}{2})}{\Gamma(q_1+q_2+q_3+n-p_0+\frac{5}{2})} e_3 = F_\lambda(0, 1) = 0. \end{aligned} \quad (3.5)$$

Hence,

$$\sum_{p_0+p_1+p_2=n-1} (-1)^{p_0} \frac{\Gamma(p_1+q_1+\frac{3}{2})\Gamma(p_2+q_2+\frac{1}{2})}{\Gamma(q_1+q_2+q_3+n-p_0+\frac{5}{2})} c_i(2p_0+1, 2p_1+1, 2p_2) = 0, \quad 0 \leq i \leq 3. \quad (3.6)$$

Similarly, taking $t_1 = 2q_1$, $t_2 = 2q_2 + 1$, $t_3 = 2q_3 + 1$, we get

$$\begin{aligned} & \sum_{p_0+p_1+p_2=n-1} \frac{c(2p_0+1, 2p_1, 2p_2+1)}{(-1)^{t_0+q_1+q_2+q_3+n+p_0+1} 2^{-2n-\lambda-2} \pi^{2n+1}} \\ & \times \frac{\Gamma(\lambda+3)\Gamma(p_1+q_1+\frac{1}{2})\Gamma(p_2+q_2+\frac{3}{2})\Gamma(q_3+\frac{3}{2})}{\Gamma(q_1+q_2+q_3+n-p_0+\frac{5}{2})} e_3 = F_\lambda(0, 1) = 0, \end{aligned} \quad (3.7)$$

which gives

$$\sum_{p_0+p_1+p_2=n-1} (-1)^{p_0} \frac{\Gamma(p_1+q_1+\frac{1}{2})\Gamma(p_2+q_2+\frac{3}{2})}{\Gamma(q_1+q_2+q_3+n-p_0+\frac{5}{2})} c_i(2p_0+1, 2p_1, 2p_2+1) = 0, \quad 0 \leq i \leq 3. \quad (3.8)$$

And taking $t_1 = 2q_1 + 1$, $t_2 = 2q_2 + 1$, $t_3 = 2q_3 + 1$, we get

$$\begin{aligned} & \sum_{p_0+p_1+p_2=n-1} \frac{c(2p_0, 2p_1+1, 2p_2+1)}{(-1)^{t_0+q_1+q_2+q_3+n+p_0} 2^{-2n-\lambda-2} \pi^{2n+1}} \\ & \times \frac{\Gamma(\lambda+3)\Gamma(p_1+q_1+\frac{3}{2})\Gamma(p_2+q_2+\frac{3}{2})\Gamma(q_3+\frac{3}{2})}{\Gamma(q_1+q_2+q_3+n-p_0+\frac{7}{2})} e_3 = F_\lambda(0, 1) = 0, \end{aligned} \quad (3.9)$$

which gives

$$\sum_{p_0+p_1+p_2=n-1} (-1)^{p_0} \frac{\Gamma(p_1+q_1+\frac{3}{2})\Gamma(p_2+q_2+\frac{3}{2})}{\Gamma(q_1+q_2+q_3+n-p_0+\frac{7}{2})} c_i(2p_0, 2p_1+1, 2p_2+1) = 0, \quad 0 \leq i \leq 3. \quad (3.10)$$

Now we solve the equations (3.3), (3.4), (3.6), (3.8) and (3.10). We would like to deal with (3.4) first. Multiplying (3.4) by $\Gamma(q_1+q_2+q_3+\frac{5}{2})$ at both sides, one gets

$$\begin{aligned} & \sum_{p_1+p_2=n} \frac{(-1)^{n+1}\Gamma(p_1+q_1+\frac{1}{2})\Gamma(p_2+q_2+\frac{1}{2})c_0(0, 2p_1, 2p_2)}{(q_1+q_2+q_3+n-1+\frac{5}{2})(q_1+q_2+q_3+n-2+\frac{5}{2}) \cdots (q_1+q_2+q_3+\frac{5}{2})} \\ & + \sum_{p_1+p_2=n-1} \frac{(-1)^n\Gamma(p_1+q_1+\frac{1}{2})\Gamma(p_2+q_2+\frac{1}{2})c_0(2, 2p_1, 2p_2)}{(q_1+q_2+q_3+n-2+\frac{5}{2})(q_1+q_2+q_3+n-3+\frac{5}{2}) \cdots (q_1+q_2+q_3+\frac{5}{2})} \\ & + \cdots + \sum_{p_1+p_2=1} \frac{\Gamma(p_1+q_1+\frac{1}{2})\Gamma(p_2+q_2+\frac{1}{2})}{q_1+q_2+q_3+\frac{5}{2}} c_0(2n-2, 2p_1, 2p_2) \\ & + (-1)\Gamma(q_1+\frac{1}{2})\Gamma(q_2+\frac{1}{2})c_0(2n, 0, 0) = \frac{2^{2n-2}}{\pi^{2n+2}} \Gamma(q_1+\frac{1}{2})\Gamma(q_2+\frac{1}{2}). \end{aligned} \quad (3.11)$$

Fix q_1 and q_2 , let $q_3 \rightarrow \infty$, we immediately get

$$c_0(2n, 0, 0) = -2^{2n-2}/\pi^{2n+2}. \quad (3.12)$$

Hence (3.11) can be reduced to

$$\begin{aligned} & \sum_{p_1+p_2=n} \frac{(-1)^{n+1}\Gamma(p_1+q_1+\frac{1}{2})\Gamma(p_2+q_2+\frac{1}{2})c_0(0, 2p_1, 2p_2)}{(q_1+q_2+q_3+n-1+\frac{5}{2})(q_1+q_2+q_3+n-2+\frac{5}{2})\cdots(q_1+q_2+q_3+\frac{5}{2})} \\ & + \sum_{p_1+p_2=n-1} \frac{(-1)^n\Gamma(p_1+q_1+\frac{1}{2})\Gamma(p_2+q_2+\frac{1}{2})c_0(2, 2p_1, 2p_2)}{(q_1+q_2+q_3+n-2+\frac{5}{2})(q_1+q_2+q_3+n-3+\frac{5}{2})\cdots(q_1+q_2+q_3+\frac{5}{2})} \\ & + \cdots + \sum_{p_1+p_2=1} \frac{\Gamma(p_1+q_1+\frac{1}{2})\Gamma(p_2+q_2+\frac{1}{2})}{q_1+q_2+q_3+\frac{5}{2}}c_0(2n-2, 2p_1, 2p_2) = 0. \end{aligned} \quad (3.13)$$

Multiplying (3.13) by $q_1+q_2+q_3+\frac{5}{2}$ and letting $q_3 \rightarrow \infty$, we get

$$\Gamma(q_1+\frac{1}{2})\Gamma(q_2+\frac{3}{2})c_0(2n-2, 0, 2) + \Gamma(q_1+\frac{3}{2})\Gamma(q_2+\frac{1}{2})c_0(2n-2, 2, 0) = 0. \quad (3.14)$$

Dividing (3.14) by $\Gamma(q_1+\frac{3}{2})$, we obtain

$$\frac{\Gamma(q_2+\frac{3}{2})}{q_1+\frac{1}{2}}c_0(2n-2, 0, 2) + \Gamma(q_2+\frac{1}{2})c_0(2n-2, 2, 0) = 0. \quad (3.15)$$

Now fix q_2 , let $q_1 \rightarrow \infty$, we immediately get

$$c_0(2n-2, 2, 0) = 0, \quad (3.16)$$

and consequently,

$$c_0(2n-2, 0, 2) = 0. \quad (3.17)$$

Repeating the above process, after (3.17) we will successively obtain

$$c_0(2p_0, 2p_1, 2p_2) = 0, \quad \forall p_0 \leq n-2, p_0+p_1+p_2=n. \quad (3.18)$$

Through similar discussions on (3.3), (3.6), (3.8) and (3.10), we can show that

$$c_i(2p_0, 2p_1, 2p_2) = 0, \quad \forall 1 \leq i \leq 3, p_0+p_1+p_2=n, \quad (3.19)$$

$$c_i(2p_0+1, 2p_1+1, 2p_2) = 0, \quad \forall 0 \leq i \leq 3, p_0+p_1+p_2=n-1, \quad (3.20)$$

$$c_i(2p_0+1, 2p_1, 2p_2+1) = 0, \quad \forall 0 \leq i \leq 3, p_0+p_1+p_2=n-1, \quad (3.21)$$

$$c_i(2p_0, 2p_1+1, 2p_2+1) = 0, \quad \forall 0 \leq i \leq 3, p_0+p_1+p_2=n-1. \quad (3.22)$$

Combining (3.12), (3.16)–(3.22), we have in fact proved that

$$c(2n, 0, 0) = -2^{2n-2}/\pi^{2n+2}, \quad c(s_0, s_1, s_2) = 0 \text{ (other cases)}.$$

So

$$s(\nu) = -2^{2n-2}/\pi^{2n+2} \frac{\partial^{2n}}{\partial x_0^{2n}}(\overline{D}N) = (2/\pi)^{2n} \frac{\partial^{2n}}{\partial x_0^{2n}}E(\nu).$$

The proof of Theorem 1.1 is complete. \square

Now we can define the Cauchy–Szegő projection operator \mathcal{C} in terms of the Cauchy–Szegő kernel:

$$(\mathcal{C}f)(q) = \lim_{\varepsilon \rightarrow 0} \int_{\partial\mathcal{U}_n} S(q + \varepsilon \mathbf{e}_0, \omega) f(\omega) d\beta(\omega), \quad \forall f \in L^2(\partial\mathcal{U}_n), \quad q \in \partial\mathcal{U}_n.$$

i.e., $\mathcal{C}(f)$ is the boundary limit of some function in $\mathcal{H}^2(\mathcal{U}_n)$, where the limit is taken in $L^2(\partial\mathcal{U}_n)$ norm. \mathcal{C} satisfies $\mathcal{C}^2 = \mathcal{C} = \mathcal{C}^*$, where \mathcal{C}^* is the adjoint operator of \mathcal{C} .

Theorem 3.3. \mathcal{C} can be extended to a bounded operator from $L^p(\partial\mathcal{U}_n)$ to $L^p(\partial\mathcal{U}_n)$, $1 < p < \infty$.

Proof. Represent \mathcal{C} by the integral on Lie group \mathcal{Q}_n , i.e., consider the following operator

$$(\mathcal{C}f)(h) = \lim_{\varepsilon \rightarrow 0} \int_{\mathcal{Q}_n} K_\varepsilon(g^{-1} \diamond h) f(g) d\beta(g), \quad \forall f \in L^2(\mathcal{Q}_n), \quad h \in \mathcal{Q}_n,$$

where $K_\varepsilon(h) = S(h(0) + \varepsilon \mathbf{e}_0, 0)$, $h = (q', t_1, t_2, t_3)$. Define the dilation on \mathcal{Q}_n by $\delta \circ (q', t_1, t_2, t_3) = (\delta q', \delta^2 t_1, \delta^2 t_2, \delta^2 t_3)$ ($\delta > 0$), and the length of the element by $\rho(h) = \max\{|q'|, |t_1|^{\frac{1}{2}}, |t_2|^{\frac{1}{2}}, |t_3|^{\frac{1}{2}}\}$, respectively. Then \mathcal{Q}_n becomes a homogeneous group of dimension $d = 4n + 6$. Consider the distribution

$$K(h) = \lim_{\varepsilon \rightarrow 0} K_\varepsilon(h) = s(\nu)|_{\nu=|q'|^2 + \mathbf{e} \cdot t},$$

and write $q' = (y_1, y_2, \dots, y_{4n})$, then one can verify that

$$\begin{aligned} |K(h)| &\leq C\rho^{-d}(h), \\ \left| \frac{\partial}{\partial y_i} K(h) \right| &\leq C\rho^{-d-1}(h), \quad 1 \leq i \leq 4n, \\ \left| \frac{\partial}{\partial t_j} K(h) \right| &\leq C\rho^{-d-2}(h), \quad 1 \leq j \leq 3. \end{aligned}$$

Because \mathcal{C} is a projection operator, \mathcal{C} must be of type (2, 2) and $\|\mathcal{C}f\|_{L^2(\partial\mathcal{U}_n)} \leq \|f\|_{L^2(\partial\mathcal{U}_n)}$. Then one can finish the proof by the theory of harmonic analysis on homogeneous groups (see, e.g. [18]). \square

Corollary 3.4. *The integral operator*

$$(\mathcal{C}f)(q) = \int_{\partial\mathcal{U}_n} S(q, \omega) f(\omega) d\beta(\omega), \quad q \in \mathcal{U}_n$$

is bounded from $L^p(\partial\mathcal{U}_n)$ to $\mathcal{H}^p(\mathcal{U}_n)$ for $1 < p < \infty$.

4 Octonionic Heisenberg group and Hardy spaces on the octonionic Siegel half space

Denote \mathbb{B} and \mathcal{U} the unit ball and Siegel half space in \mathbb{O}^2 respectively, viz.,

$$\mathbb{B} = \{(\sigma_1, \sigma_2) \in \mathbb{O}^2 : |\sigma_1|^2 + |\sigma_2|^2 < 1\},$$

$$\mathcal{U} = \{(\tau_1, \tau_2) \in \mathbb{O}^2 : \operatorname{Re} \tau_2 > |\tau_1|^2\}.$$

Then one can verify that the Cayley transform

$$\sigma_1 = \frac{2\tau_1(1 + \overline{\tau_2})}{|1 + \tau_2|^2}, \quad \sigma_2 = \frac{(1 + \overline{\tau_2})(1 - \tau_2)}{|1 + \tau_2|^2} = \frac{(1 - \tau_2)(1 + \overline{\tau_2})}{|1 + \tau_2|^2}$$

is a bijection from \mathcal{U} to \mathbb{B} with the inversion being given by

$$\tau_1 = \frac{\sigma_1(1 + \overline{\sigma_2})}{|1 + \sigma_2|^2}, \quad \tau_2 = \frac{(1 + \overline{\sigma_2})(1 - \sigma_2)}{|1 + \sigma_2|^2} = \frac{(1 - \sigma_2)(1 + \overline{\sigma_2})}{|1 + \sigma_2|^2}.$$

Unlike the case of several complex variables, the above Cayley transform is neither left \mathbb{O} -analytic nor right \mathbb{O} -analytic.

The boundary of \mathcal{U} is denoted by $\partial\mathcal{U} = \{(\tau_1, \tau_2) \in \mathbb{O}^2 : \operatorname{Re} \tau_2 = |\tau_1|^2\}$. We introduce three mappings on \mathcal{U} here: dilations, rotations and translations. Let $\tau = (\tau_1, \tau_2) \in \mathbb{O}^2$, for every positive number δ , define the dilation $\delta \circ \tau$ as follows:

$$\delta \circ \tau = \delta \circ (\tau_1, \tau_2) = (\delta\tau_1, \delta^2\tau_2),$$

it is non-isotropic due to the structure of \mathcal{U} . For each rotation \mathcal{R} on \mathbb{O} , define the rotation $\mathcal{R}(\tau)$ by

$$\mathcal{R}(\tau) = \mathcal{R}(\tau_1, \tau_2) = (\mathcal{R}(\tau_1), \tau_2).$$

Obviously, both the dilation and rotation give self mappings of \mathcal{U} that can be extended to mappings on the boundary $\partial\mathcal{U}$, but in general they are neither left \mathbb{O} -analytic nor right \mathbb{O} -analytic.

Before we describe the translations on \mathcal{U} , we introduce the octonionic Heisenberg group, denoted by \mathcal{O} . This group consists of the set

$$\mathbb{O} \times \mathbb{R}^7 = \{[\omega, t] = [\omega, t_1, \dots, t_7] : \omega \in \mathbb{O}, t = (t_1, \dots, t_7) \in \mathbb{R}^7\}$$

with the multiplication law

$$[\alpha, t] \diamond [\beta, s] = [\alpha + \beta, t_1 + s_1 + 2\operatorname{Im}_1(\overline{\alpha}\beta), \dots, t_7 + s_7 + 2\operatorname{Im}_7(\overline{\alpha}\beta)], \quad (4.1)$$

which makes $\mathbb{O} \times \mathbb{R}^7$ into Lie group with the neutral element $[0, 0]$ and the inverse element of $[\omega, t]$ being given by $[\omega, t]^{-1} = [-\omega, -t]$.

For each element $[\omega, t] \in \mathcal{O}$, we define the translation on \mathcal{U} :

$$[\omega, t] : (\tau_1, \tau_2) \mapsto (\tau_1 + \omega, \tau_2 + |\omega|^2 + 2\overline{\omega}\tau_1 + \mathbf{e} \cdot t), \quad (4.2)$$

where $\mathbf{e} = (e_1, \dots, e_7)$, $\mathbf{e} \cdot t = \sum_{i=1}^7 e_i t_i$. This mapping preserves the function $r(\tau) = \operatorname{Re} \tau_2 - |\tau_1|^2$, hence it maps $\mathcal{U} = \{\tau : r(\tau) > 0\}$ to itself and preserves the boundary $\partial\mathcal{U} = \{\tau : r(\tau) = 0\}$.

Besides, the reader can check that the mapping (4.2) defines an action of the group \mathcal{O} on \mathcal{U} . Via this action at the origin

$$[\omega, t] : (0, 0) \mapsto (\omega, |\omega|^2 + \mathbf{e} \cdot t),$$

we can identify \mathcal{O} with $\partial\mathcal{U} : \mathcal{O} \ni [\omega, t] \mapsto (\omega, |\omega|^2 + \mathbf{e} \cdot t) \in \partial\mathcal{U}$.

Let dh be the Haar measure on \mathcal{O} , using the identification of $\partial\mathcal{U}$ with \mathcal{O} we introduce the measure $d\beta$ on $\partial\mathcal{U}$ by the following integral identity:

$$\int_{\partial\mathcal{U}} F(\tau) d\beta(\tau) = \int_{\mathbb{O} \times \mathbb{R}^7} F(\tau_1, |\tau_1|^2 + \mathbf{e} \cdot t) d\tau_1 dt, \quad \forall F(\tau) \in C_c(\partial\mathcal{U}),$$

where $C_c(\partial\mathcal{U})$ is the set of continuous functions with compact support on $\partial\mathcal{U}$. With this measure we can define the space $L^p(\mathcal{O}) = L^p(\partial\mathcal{U})$ ($0 < p < \infty$).

For any function $F(\tau)$ defined on \mathcal{U} , $F_\varepsilon(\tau) = F(\tau + \varepsilon \mathbf{e}_0)$ is called the vertical translate of $F(\tau)$, here $\mathbf{e}_0 = (0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0)$. Obviously, if $\varepsilon > 0$, then $F_\varepsilon(\tau)$ is well defined in some neighborhood of $\partial\mathcal{U}$. In particular, $F_\varepsilon(\tau)$ is well defined on $\partial\mathcal{U}$. For $\frac{6}{7} < p < \infty$, we define the

Hardy space $\mathcal{H}^p(\mathcal{U})$ on the octonionic Siegel half space to be the set of functions $F(\tau)$ which are left \mathbb{O} -analytic on \mathcal{U} with respect to τ_1, τ_2 respectively, and satisfy

$$\|F\|_{\mathcal{H}^p(\mathcal{U})} := \left(\sup_{\varepsilon > 0} \int_{\partial\mathcal{U}} |F_\varepsilon(\tau)|^p d\beta(\tau) \right)^{1/p} < \infty.$$

$\mathcal{H}^p(\mathcal{U})$ is a Banach space when $1 \leq p < \infty$, a Fréchet space when $\frac{6}{7} < p < 1$. In particular, $\mathcal{H}^2(\mathcal{U})$ becomes a real-linear Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ being defined by (it will be seen later that the definition is reasonable):

$$\langle F, G \rangle = \lim_{\varepsilon \rightarrow 0} \int_{\partial\mathcal{U}} \overline{G_\varepsilon(\tau)} F_\varepsilon(\tau) d\beta(\tau), \quad \forall F, G \in \mathcal{H}^2(\mathcal{U}).$$

Our main result concerning $\mathcal{H}^p(\mathcal{U})$ is:

Theorem 4.1. *Suppose $F(\tau) \in \mathcal{H}^p(\mathcal{U})$ ($\frac{6}{7} < p < \infty$), then*

- (i) *There exists an $F^b \in L^p(\partial\mathcal{U})$, such that $F_\varepsilon(\tau)|_{\partial\mathcal{U}} \rightarrow F^b$ ($\varepsilon \rightarrow 0$) in the sense of $L^p(\partial\mathcal{U})$ norm and almost everywhere.*
- (ii) *$\{F^b\}$ is closed subspace of $L^p(\partial\mathcal{U})$. Moreover,*
- (iii) *$\|F^b\|_{L^p(\partial\mathcal{U})} = \|F\|_{\mathcal{H}^p(\mathcal{U})}$.*

Its proof is based on the Subharmonicity of powers of \mathbb{O} -analytic functions and some lemmas concerning the Hardy space $\mathcal{H}^p(\mathbb{R}_+^8)$.

Denote the upper half space $\{\tau = u + \mathbf{e} \cdot v : u > 0\} \subset \mathbb{R}^8$ by \mathbb{R}_+^8 . The Hardy space $\mathcal{H}^p(\mathbb{R}_+^8)$ ($\frac{6}{7} < p < \infty$) is defined to be the set of functions $f(\tau)$ which are left \mathbb{O} -analytic on \mathbb{R}_+^8 with respect to $\tau = u + \mathbf{e} \cdot v$ and satisfy $\|f\|_{\mathcal{H}^p(\mathbb{R}_+^8)} := \left(\sup_{u>0} \int_{\mathbb{R}^7} |f(u + \mathbf{e} \cdot v)|^p dv \right)^{\frac{1}{p}} < \infty$.

Lemma 4.2. [1] *Suppose $f \in C^1(\Omega, \mathbb{O})$ satisfies $Df = 0$, then $|f|^p$ is subharmonic in Ω when $p \geq \frac{6}{7}$.*

By analogous discussions as in [6, 9], one can prove that

Lemma 4.3. *For any $f \in \mathcal{H}^p(\mathbb{R}_+^8)$ ($\frac{6}{7} < p < \infty$), there holds*

$$\int_{\mathbb{R}^7} \sup_{u>0} |f(u + \mathbf{e} \cdot v)|^p dv \leq C_p \|f\|_{\mathcal{H}^p(\mathbb{R}_+^8)}^p. \quad (4.3)$$

And there exists an $f^b \in L^p(\mathbb{R}^7)$, such that

$$f(u + \mathbf{e} \cdot v) \rightarrow f^b(v) \quad (u \rightarrow 0), \quad \text{for a.e. } v \in \mathbb{R}^7, \quad (4.4)$$

$$\lim_{u \rightarrow 0} \int_{\mathbb{R}^7} |f(u + \mathbf{e} \cdot v) - f^b(v)|^p dv = 0. \quad (4.5)$$

Moreover,

$$\|f^b\|_{L^p(\mathbb{R}^7)} = \|f\|_{\mathcal{H}^p(\mathbb{R}_+^8)}. \quad (4.6)$$

Lemma 4.4. *For each $F \in \mathcal{H}^p(\mathcal{U})$ ($\frac{6}{7} < p < \infty$), $\tau_1 \in \mathbb{O}$ and $\delta > 0$, we have*

$$f(\tau_2) := F(\tau_1, \tau_2 + \delta + |\tau_1|^2) \in \mathcal{H}^p(\mathbb{R}_+^8).$$

Remark. For $\mathcal{H}^2(\mathcal{U}^n)$ with several complex variables or $\mathcal{H}^p(\mathcal{U}_n)$ with several quaternionic variables, the proof of the lemma corresponding to Lemma 4.4 is to consider the special case when $\tau_1 = 0$ and $\delta > 0$ first, then prove the general case through the action of Heisenberg group on the Siegel half space. However, this method is not valid for Lemma 4.4, since a left (right) \mathbb{O} -analytic function in general does not preserve the analyticity after left (right) multiplying the variable by an octonion constant. In fact, we have

Proposition 4.5. *Suppose $F(q_1, q_2)$ is left \mathbb{H} -regular on $\mathbb{H} \times \mathbb{H}$ with respect to q_1 and q_2 respectively, then for any quaternionic constant α , $f(q_1) := F(q_1, \alpha q_1)$ is left \mathbb{H} -regular on \mathbb{H} with respect to q_1 .*

Proposition 4.6. *Suppose $F(\tau_1, \tau_2)$ is left \mathbb{O} -analytic on $\mathbb{O} \times \mathbb{O}$ with respect to τ_1 and τ_2 respectively, then the following two conditions are equivalent:*

- (i) *For any octonionic constant α , $f(\tau_1) := F(\tau_1, \alpha \tau_1)$ is left \mathbb{O} -analytic on \mathbb{O} with respect to τ_1 .*
- (ii) *For each fixed $\tau_1 \in \mathbb{O}$, all components of $g(\tau_2) := \overline{F(\tau_1, \tau_2)}$ consist a Stein–Weiss conjugate harmonic system on \mathbb{R}^8 .*

Proposition 4.5 can be verified by direct computation, Proposition 4.6 is a corollary of the following:

Proposition 4.7. $\forall \alpha \in \mathbb{O}$, $f(\alpha x)$ is left \mathbb{O} -analytic on $\mathbb{O} \iff \forall \alpha \in \mathbb{O}$, $f(x\alpha)$ is right \mathbb{O} -analytic on $\mathbb{O} \iff$ All components of $f(x)$ consist a Stein–Weiss conjugate harmonic system on \mathbb{R}^8

Proof. Denote $f(x) = \sum_{j=0}^7 f_j e_j$, $\alpha = \sum_{l=0}^7 \alpha_l e_l$, $y(x) = \alpha x = \sum_{k=0}^7 y_k(x) e_k$, then $y_k = \text{Re}((\alpha x) \overline{e_k})$, $\frac{\partial y_k}{\partial x_i} = \text{Re}((\alpha e_i) \overline{e_k}) = \sum_{l=0}^7 \alpha_l \text{Re}((e_l e_i) \overline{e_k})$. Thus we have

$$\begin{aligned}
D(f(\alpha x)) &= \sum_{i,j} e_i e_j \frac{\partial f_j}{\partial x_i} \\
&= \sum_{i,j,k} e_i e_j \frac{\partial f_j}{\partial y_k} \frac{\partial y_k}{\partial x_i} \\
&= \sum_{i,j,k,l} \alpha_l e_i e_j \frac{\partial f_j}{\partial y_k} \text{Re}((e_l e_i) \overline{e_k}) \\
&= \sum_{j,k,l} \alpha_l (\varepsilon \overline{e_l} e_k) e_j \frac{\partial f_j}{\partial y_k} \text{Re}((e_l (\varepsilon \overline{e_l} e_k)) \overline{e_k}) \quad (\varepsilon = 1 \text{ or } -1) \\
&= \sum_{j,k,l} \varepsilon^2 \alpha_l (\overline{e_l} e_k) e_j \frac{\partial f_j}{\partial y_k} \\
&= \sum_{j,k} (\overline{\alpha} e_k) e_j \frac{\partial f_j}{\partial y_k} \\
&= \sum_{j,k} \overline{\alpha} (e_k e_j) \frac{\partial f_j}{\partial y_k} + \sum_{j,k} [\overline{\alpha}, e_k, e_j] \frac{\partial f_j}{\partial y_k} \\
&= \overline{\alpha} Df(y) + \sum_{1 \leq j \neq k \leq 7} [\overline{\alpha}, e_k, e_j] \frac{\partial f_j}{\partial y_k} \\
&= \overline{\alpha} Df(y) + \sum_{1 \leq j < k \leq 7} \left(\frac{\partial f_j}{\partial y_k} - \frac{\partial f_k}{\partial y_j} \right) [\overline{\alpha}, e_k, e_j],
\end{aligned}$$

from which it is not difficult to get (cf. [16])

$$\begin{aligned} & \forall \alpha \in \mathbb{O}, D(f(\alpha x)) = 0 \\ \iff & \frac{\partial f_0}{\partial x_0} = \sum_{i=1}^7 \frac{\partial f_i}{\partial x_i}, \quad \frac{\partial f_0}{\partial x_i} = -\frac{\partial f_i}{\partial x_0} \quad (1 \leq i \leq 7) \quad \text{and} \quad \frac{\partial f_j}{\partial x_k} = \frac{\partial f_k}{\partial x_j} \quad (1 \leq j < k \leq 7). \end{aligned}$$

□

Proof of Lemma 4.4. Write $\tau_2 = u + \mathbf{e} \cdot v$, for any $\omega \in \mathbb{O}$, applying Lemma 4.2 to the function

$$f(u + \mathbf{e} \cdot v) = F(\omega, u + \delta + |\omega|^2 + \mathbf{e} \cdot v), \quad u > 0,$$

we get

$$|f(u + \mathbf{e} \cdot v)|^p \leq c_{\omega, \delta} \int_{|\tau_1| < \alpha, |\tau_2| < \delta/2} |F(\tau_1 + \omega, \tau_2 + u + \delta + |\omega|^2 + \mathbf{e} \cdot v)|^p d\tau_1 d\tau_2, \quad (4.7)$$

where $\alpha > 0$ satisfies $\alpha^2 + 2\alpha|\omega| = \delta/2$, and $c_{\omega, \delta}^{-1}$ is the volume of the cuboid $\{(\tau_1, \tau_2) \in \mathbb{O}^2 : |\tau_1| < \alpha, |\tau_2| < \delta/2\}$. Since $|\operatorname{Re} \tau_2| < \delta/2$ and $u > 0$, we have $\operatorname{Re}(\tau_2 + u + \delta + |\omega|^2) \in (\delta/2 + |\omega|^2, 3\delta/2 + |\omega|^2)$, thus $\operatorname{Re}(\tau_2 + u + \delta + |\omega|^2) > |\tau_1 + \omega|^2$, which guarantees that the integration in (4.7) makes sense. Write $\tau_2 = x + \mathbf{e} \cdot y$ and integrate (4.7) on both sides with respect to v over \mathbb{R}^7 , after applying the Fubini theorem, we deduce that

$$\begin{aligned} & \int_{\mathbb{R}^7} |f(u + \mathbf{e} \cdot v)|^p dv \\ & \leq c_{\omega, \delta} \int_{|\tau_1| < \alpha} \int_{\mathbb{R}^7} \int_{|\tau_2| < \delta/2} |F(\tau_1 + \omega, \tau_2 + u + \delta + |\omega|^2 + \mathbf{e} \cdot v)|^p d\tau_1 dv d\tau_2 \\ & \leq c'_{\omega, \delta} \int_{|\tau_1| < \alpha} \int_{\mathbb{R}^7} \int_{|x| < \delta/2} |F(\tau_1 + \omega, x + u + \delta + |\omega|^2 + \mathbf{e} \cdot y)|^p d\tau_1 dy dx. \end{aligned}$$

Making change of variables $x + u + \delta + |\omega|^2 = \varepsilon + |\tau_1 + \omega|^2$. Since $|x| < \delta/2$ and $|\tau_1 + \omega|^2 < \delta/2 + |\omega|^2$, the range of the new variable ε lies in the interval $(u, u + 3\delta/2 + |\omega|^2)$. So the last integral is majorized by

$$\begin{aligned} & c'_{\omega, \delta} \int_u^{u+3\delta/2+|\omega|^2} \int_{|\tau_1| < \alpha} \int_{\mathbb{R}^7} |F(\tau_1 + \omega, \varepsilon + |\tau_1 + \omega|^2 + \mathbf{e} \cdot y)|^p d\tau_1 dy d\varepsilon \\ & \leq c'_{\omega, \delta} (3\delta/2 + |\omega|^2) \|F\|_{\mathcal{H}^p(\mathcal{U})}^p < \infty. \end{aligned}$$

i.e., $f(u + \mathbf{e} \cdot v) = F(\omega, u + \delta + |\omega|^2 + \mathbf{e} \cdot v) \in \mathcal{H}^p(\mathbb{R}_+^8)$. □

Proof of Theorem 4.1. Applying the maximal inequality (4.3) and equality (4.6) to the function $f(\tau_2) = F(\tau_1, \tau_2 + \delta + |\tau_1|^2) (\tau_2 = x + \mathbf{e} \cdot y)$, we obtain

$$\int_{\mathbb{R}^7} \sup_{x>0} |F(\tau_1, x + \delta + |\tau_1|^2 + \mathbf{e} \cdot y)|^p dy \leq C_p \int_{\mathbb{R}^7} |F(\tau_1, \delta + |\tau_1|^2 + \mathbf{e} \cdot y)|^p dy,$$

integrate it over $\tau_1 \in \mathbb{O}$ and write $x = \varepsilon$, one gets

$$\int_{\partial \mathcal{U}} \sup_{\varepsilon>0} |F(\tau + (\varepsilon + \delta)\mathbf{e}_0)|^p d\beta(\tau) \leq C_p \int_{\partial \mathcal{U}} |F(\tau + \delta\mathbf{e}_0)|^p d\beta(\tau).$$

Letting $\delta \rightarrow 0$ on both sides, by Fatou's lemma we see that

$$\int_{\partial \mathcal{U}} \sup_{\varepsilon>0} |F(\tau + \varepsilon\mathbf{e}_0)|^p d\beta(\tau) \leq C_p \|F\|_{\mathcal{H}^p(\mathcal{U})}^p < \infty.$$

We conclude from the above inequality that for almost all $\tau_1 \in \mathbb{O}$, the function $F(\tau_1, \tau_2 + |\tau_1|^2)$, as a function of τ_2 , belongs to $\mathcal{H}^p(\mathbb{R}_+^8)$. Thus by (4.4) we see that the limit $\lim_{\varepsilon \rightarrow 0} F(\tau + \varepsilon \mathbf{e}_0) = F^b(\tau)$ exists for almost every $\tau \in \partial\mathcal{U}$. Note that

$$\int_{\partial\mathcal{U}} |F_\varepsilon(\tau) - F^b(\tau)|^p d\beta(\tau) \leq 2^p \int_{\partial\mathcal{U}} \sup_{\varepsilon > 0} |F_\varepsilon(\tau)|^p d\beta(\tau) \leq 2^p C_p \|F\|_{\mathcal{H}^p(\mathcal{U})}^p < \infty,$$

by Lebesgue's dominated convergence theorem we arrive at

$$\lim_{\varepsilon \rightarrow 0} \int_{\partial\mathcal{U}} |F_\varepsilon(\tau) - F^b(\tau)|^p d\beta(\tau) = 0.$$

Now we show the property (iii). By Fatou's lemma, it follows that

$$\int_{\partial\mathcal{U}} |F^b(\tau)|^p d\beta(\tau) \leq \sup_{\varepsilon > 0} \int_{\partial\mathcal{U}} |F(\tau + \varepsilon \mathbf{e}_0)|^p d\beta(\tau) = \|F\|_{\mathcal{H}^p(\mathcal{U})}^p. \quad (4.8)$$

On the other hand, (4.6) leads to

$$\begin{aligned} \int_{\mathbb{R}^7} |F(\tau_1, \varepsilon + |\tau_1|^2 + \mathbf{e} \cdot y)|^p dy &\leq \sup_{\varepsilon > 0} \int_{\mathbb{R}^7} |F(\tau_1, \varepsilon + |\tau_1|^2 + \mathbf{e} \cdot y)|^p dy \\ &= \int_{\mathbb{R}^7} |F^b(\tau_1, |\tau_1|^2 + \mathbf{e} \cdot y)|^p dy, \end{aligned}$$

integrating it over $\tau_1 \in \mathbb{O}$ on both sides and taking supremum over $\varepsilon > 0$, combining (4.8) we get property (iii).

To prove the assertion (ii), one can deduce from the inequality (4.7) that for any compact set $\mathbb{K} \subset \mathcal{U}$, there always holds

$$\sup_{\tau \in \mathbb{K}} |F(\tau)| \leq C_{\mathbb{K}, p} \|F\|_{\mathcal{H}^p(\mathcal{U})},$$

from which we conclude that if a sequence $\{F_n\}$ converges in $\mathcal{H}^p(\mathcal{U})$ norm (or metric), then it must converge uniformly on any compact subset of \mathcal{U} , which implies that the space $\mathcal{H}^p(\mathcal{U})$ is complete with respect to its norm (or metric). The proof of Theorem 4.1 is complete. \square

Remark. Similarly, we can generalize Theorem 4.1 to \mathbb{O}^n . However, octonions is non-associative, up to now, we are still not sure about the existence of the octonionic Cauchy–Szegő kernel, or how to derive the exact form of the kernel provided that it exists?

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